Institute of Applied Mechanics and Mechatronics,
Faculty of Mechanical Engineering STU Bratislava; http://www.sjf.stuba.sk

2-5532 Theory of Mechanisms

Applied Mechanics and Mechatronics for bachelor study, year 3, summer sem.
Guarantee: Assoc.Prof. František Palčák, PhD., ÚAMM 02010

Lecture 4: Cauchy's-Poisson's decomposition of general planar motion of a part in the multibody system

Sections in Lecture 4:

S1 Cauchy's-Poisson's decomposition of general planar motion of the body to the fictive translation represented by reference point and to the fictive rotation about reference point.

S2 Development of general formula for time derivation of the vector expressed in different spaces.

S3 Equations of centrodes k_P, k_H

S1 Cauchy's decomposition of general planar motion of the body to the fictive translation represented by reference point and to the fictive rotation about reference point.

Position

The position of the point B_1 from PAR 3 (coupler in the piston–crank mechanism) wrt PAR 1 (ground) is given by vector equation

$$\overline{r}_{B31} = \overline{r}_{A31} + \overline{r}_{BA} \tag{1}$$

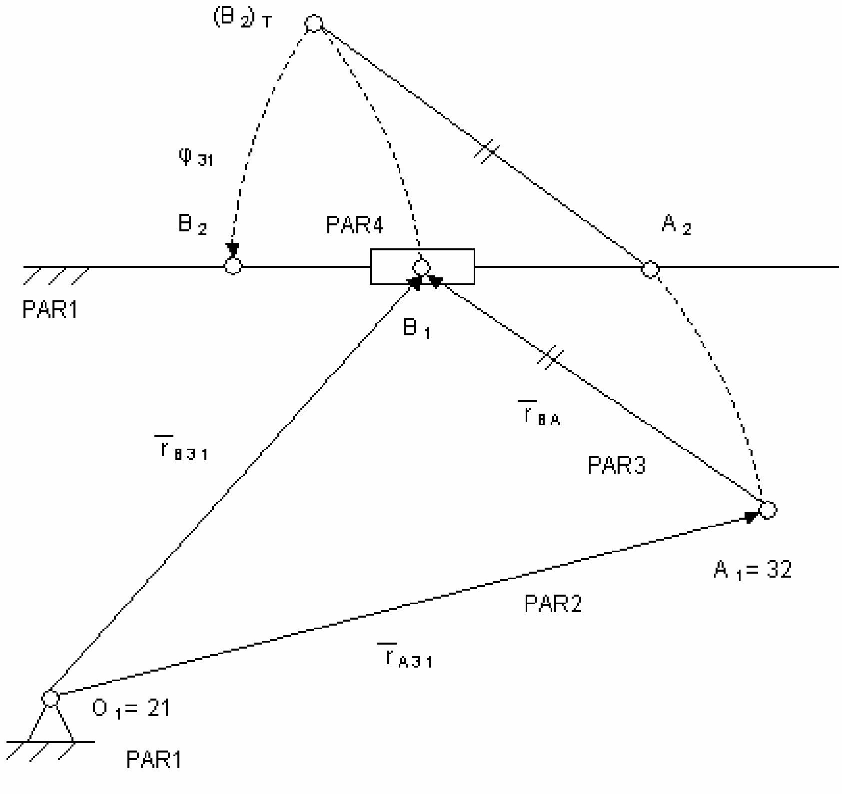

Fig.1 Graphical depiction of Cauchy's-Poisson's decomposition of general planar motion of a body in the multibody system

Velocity

In generally, the time derivative of radius vector, expressed in the space $\{a\}$, in the same space $\{a\}$, is velocity vector expressed in the space $\{a\}$. In the equation (1): $\overline{r}_{B31} = \overline{r}_{A31} + \overline{r}_{BA}$ all vector quantities are expressed wrt reference space of the frame 1. Then by time derivative of equation (1) $[\overline{r}_{B31}]_1^{\bullet} = [\vec{r}_{A31}]_1^{\bullet} + [\vec{r}_{BA}]_1^{\bullet}$ we obtain vector equation (2) for instantaneous velocities

$$\overline{v}_{B31} = \overline{v}_{A31} + \overline{v}_{BA31} \qquad (2)$$

Equation (2) is known as Cauchy's (1827)-Poisson's (1834) decomposition of general planar motion 3/1 to the fictive translation 3/1, represented by reference point A, when abscissa $\overline{AB}$ is displaced from its initial position $\overline{A_1B_1}$ to the intermittent position $\overline{A_2(B_2)_T}$ and then is displaced from the position $\overline{A_2(B_2)_T}$ to the final position $\overline{A_2B_2}$ by the fictive rotation 3/1 about reference point A.
Time derivative of rotating position vector $\overline{r}_{BA}$ expressed in the space $\{3\}$ is Euler's instantaneous circumference velocity

$$\overline{v}_{BA31} = \overline{\omega}_{31} \times \overline{r}_{BA} \qquad (3)$$

of the point B wrt A during fictive rotation of $\overline{A_2(B_2)_T}$ about reference point A in the position A_2.

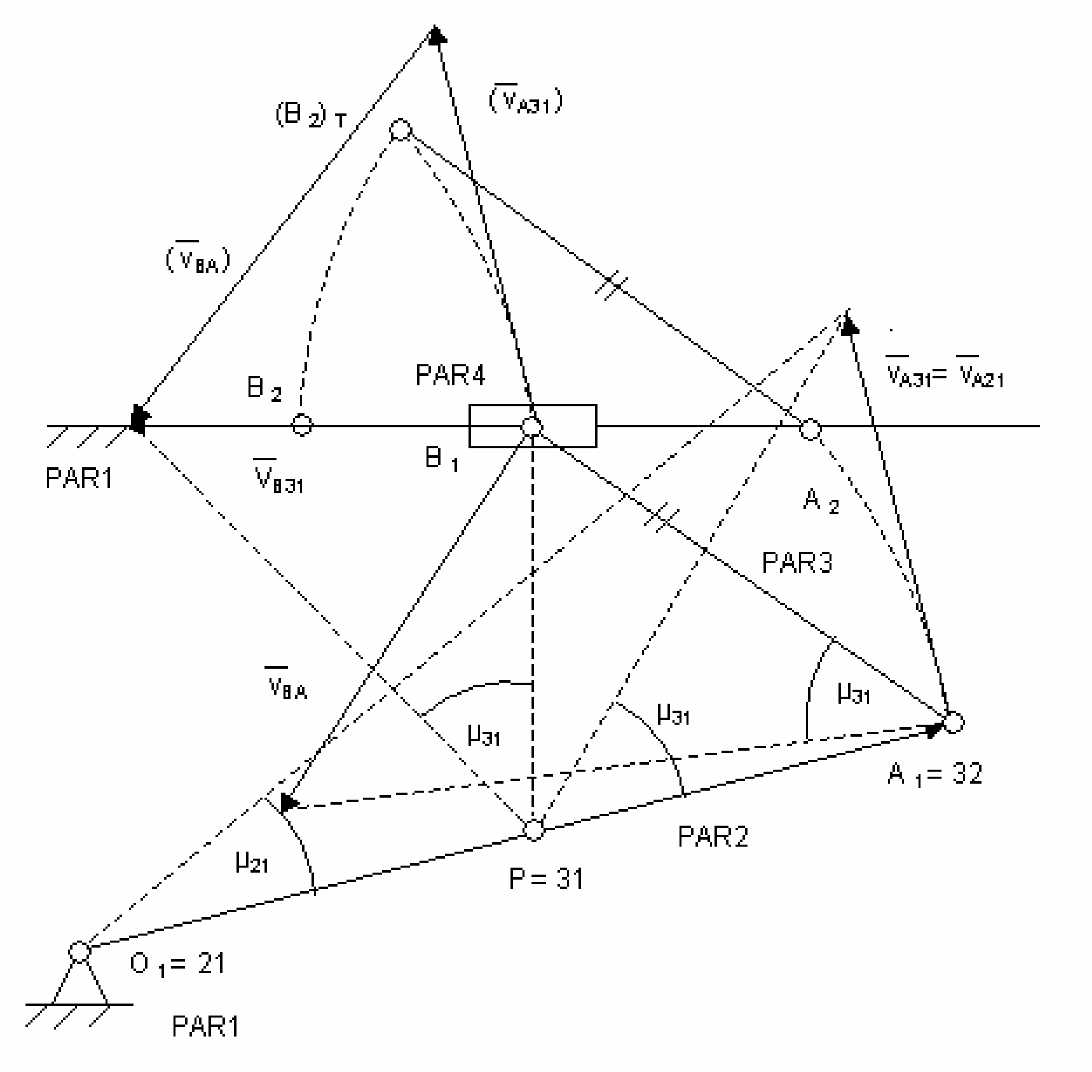

Fig.2 Graphical construction of vector equation $\overline{v}_{B31} = \overline{v}_{A31} + \overline{v}_{BA31}$.

Graphical method For given velocity $\overline{v}_{A31}$ we can apply the graphical construction of vector equation $\overline{v}_{B31} = \overline{v}_{A31} + \overline{v}_{BA31}$.

Acceleration Acceleration $\overline{a}_{B31}$ of the point B_1 we obtain by time derivation of velocity equation $[\overline{v}_{B31}]_1^{\bullet} = [\overline{v}_{A31}]_1^{\bullet} + \frac{d}{dt}(\overline{\omega}_{31} \times \overline{r}_{BA})$

$$\overline{a}_{B31} = \overline{a}_{A31} + \overline{\alpha}_{31} \times \overline{r}_{BA} + \overline{\omega}_{31} \times \overline{v}_{BA} \quad (4)$$

S2 Development of general formula for time derivation of the vector expressed in different spaces.

Position of the P

The coupler PAR3 from slider crank mechanism on Fig.3 is performing general motion 3/1 wrt PAR1. In the instantaneous initial position $\overline{A_1B_1}$ the point C from coupler PAR3, $C \in 3$ coincident with instantaneous slew centre $C \equiv 31$ has instantaneous zero velocity $\bar{v}_{C31} = \bar{0}$. The radius vector of position of the virtual point P (instantaneous slew centre 31) wrt origin O_1 of global coordinate system representing the PAR1 is given by vector equation

$$\bar{r}_{P1} = \bar{r}_{A31} + \bar{r}_{P3} \tag{1}$$

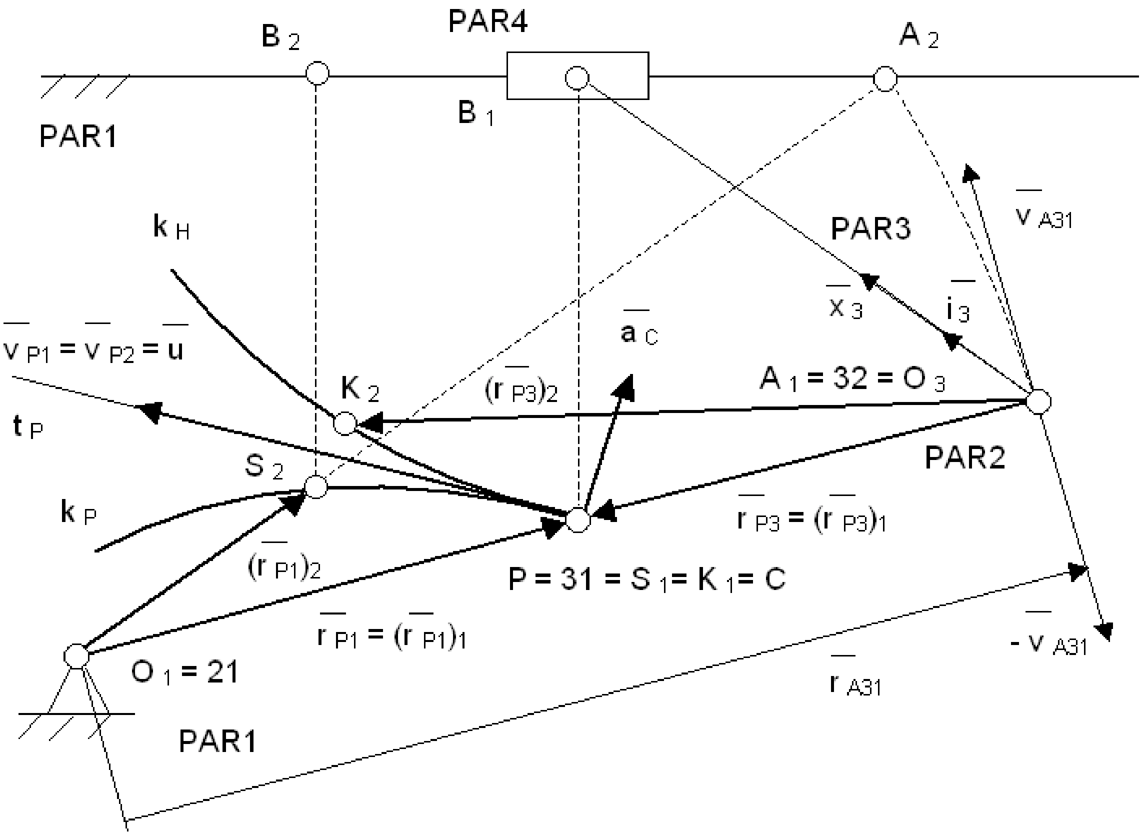

Fig.3 Slider crank mechanism with fixed k_P, resp. movable k_H centrode.

Velocity of the P

When we want to obtain the instantaneous velocity of virtual point P during displacement of virtual point P along fixed k_P, resp. movable k_H centrode, it is necessary to differentiate equation (1) in the space $\{1\}$:

$$[\bar{r}_{P1}]^{\cdot}_1 = [\vec{r}_{A31}]^{\cdot}_1 + [\bar{r}_{P3}]^{\cdot}_1 \tag{2}$$

The radius vector $\overline{r}_{P3}$ is expressed in the space $\{3\}$

$$\overline{r}_{P3} = (\overline{r}_{P3} \cdot \overline{i}_3)\overline{i}_3 + (\overline{r}_{P3} \cdot \overline{j}_3)\overline{j}_3 \qquad (3)$$

so time derivative $[\overline{r}_{P3}]^{\bullet}_{1}$ of radius vector $\overline{r}_{P3}$ in different space $\{1\}$ requires a development of a general rule. Let us denote

$$r_{P3x} = \overline{r}_{P3} \cdot \overline{i}_3 \qquad (4)$$

$$r_{P3y} = \overline{r}_{P3} \cdot \overline{j}_3 \qquad (5)$$

the coordinates of the radius vector $\overline{r}_{P3}$. Time derivative of the coordinates r_{P3x}, and r_{P3y} of radius vector $\overline{r}_{P3}$ are coordinates v_{P3x}, and v_{P3y} of instantaneous velocity $\overline{v}_{P3}$ of point P resp.

$$v_{P3x} = \frac{d}{dt} r_{P3x} \qquad (6)$$

$$v_{P3y} = \frac{d}{dt} r_{P3y} \qquad (7)$$

then we obtain

$$[\overline{r}_{P3}]^{\bullet}_{1} = (\frac{d}{dt} r_{P3x})\overline{i}_3 + r_{P3x}\frac{d\overline{i}_3}{dt} + (\frac{d}{dt} r_{P3y})\overline{j}_3 + r_{P3y}\frac{d\overline{j}_3}{dt} \qquad (8)$$

Differentiation of unit vector $\overline{i}_3$, resp. $\overline{j}_3$ rotating by angular velocity $\overline{\omega}_{31}$ wrt time is equal to the cross product

$$\frac{d\overline{i}_3}{dt} = \overline{\omega}_{31} \times \overline{i}_3 \qquad (9)$$

resp.

$$\frac{d\overline{j}_3}{dt} = \overline{\omega}_{31} \times \overline{j}_3 \qquad (10)$$

Then we can write the equation (8) in the form

$$[\overline{r}_{P3}]^{\bullet}_{1} = [\overline{r}_{P3}]^{\bullet}_{3} + \overline{\omega}_{31} \times \overline{r}_{P3} \qquad (11)$$

Forasmuch as

$$\left[\bar{r}_{P3}\right]^{\bullet}_{3} = \bar{v}_{P3} \tag{12}$$

and as can be seen on Fig.1

$$\bar{\omega}_{31} \times \bar{r}_{P3} = -\bar{v}_{A31} \tag{13}$$

after substituting (13) into equation (2) we obtain vector equation for velocities

$$\bar{v}_{P1} = \bar{v}_{A31} + \bar{v}_{P3} - \bar{v}_{A31} \tag{14}$$

from which yield that velocities $\bar{v}_{P1}$, resp. $\bar{v}_{P3}$ of virtual point P displacement along fixed, resp. movable centrodes wrt space $\{1\}$ are equal.

$$\bar{v}_{P1} = \bar{v}_{P3} = \bar{u} \tag{15}$$

The line of action of the velocity $\bar{u}$ is a tangent t_P of centrodes k_P and k_H.

Generalization

Task for development of time derivative of a vector quantity $\bar{r}_{P\,a}$ in different space $\{b\}$like the space $\{a\}$ in which is expressed, can be generalized according to the Equation (11) $\left[\bar{r}_{P3}\right]^{\bullet}_{1} = \left[\bar{r}_{P3}\right]^{\bullet}_{3} + \bar{\omega}_{31} \times \bar{r}_{P3}$ into form

$$\left[\bar{r}_{P\,a}\right]^{\bullet}_{b} = \left[\bar{r}_{P\,a}\right]^{\bullet}_{a} + \bar{\omega}_{ab} \times \bar{r}_{P\,a} \tag{16}$$

where the time derivative of vector quantity $\bar{r}_{P\,a}$ in different space $\{b\}$ like the space $\{a\}$ in which is expressed, is a sum of time derivative of vector quantity $\bar{r}_{P\,a}$ in the same space $\{a\}$ and cross product of angular velocity between spaces $\{a\},\{b\}$ with vector quantity $\bar{r}_{P\,a}$ expressed in the space $\{a\}$.

S3 Equations of k_P, k_H

Equations of k_P, k_H

Let us apply the equation $\bar{v}_{B31} = \bar{v}_{A31} + \bar{v}_{BA31}$ of the Cauchy's-Poisson's decomposition of general planar motion 3/1 for point $C \in 3$:

$$\bar{v}_{C31} = \bar{v}_{A31} + \bar{v}_{CA31} \tag{17}$$

Taking into account that $C \equiv 31$, the instantaneous velocity

$$\bar{v}_{C31} = \bar{0} \tag{18}$$

and the Euler's instantaneous circumference velocity $\bar{v}_{CA31}$ of fictive rotation of radius vector $\bar{r}_{P3}$ about reference point A

$$\bar{v}_{CA31} = \bar{\omega}_{31} \times \bar{r}_{P3} \tag{19}$$

Cross product of the Eq. (17) from left side by $\bar{\omega}_{31}$ is then

$$\bar{0} = \bar{\omega}_{31} \times \bar{v}_{A31} + \bar{\omega}_{31} \times (\bar{\omega}_{31} \times \bar{r}_{P3}) \tag{20}$$

After realisation of double cross product we can isolate the radius vector $\bar{r}_{P3}$

$$\bar{r}_{P3} = \frac{\bar{\omega}_{31} \times \bar{v}_{A31}}{\omega_{31}^{2}} \tag{21}$$

When all vector quantities in the Eq.21 are expressed in the space $\{3\}$, then locus of end points of the radius vector $\bar{r}_{P3}$ is movable centrode k_H (see Fig.1) and Eq.21 is equation of the movable centrode k_H

$$\bar{r}_{P3} = \frac{1}{\omega_{31}^{2}} \begin{vmatrix} \bar{i}_3 & \bar{j}_3 & \bar{k}_3 \\ 0 & 0 & \bar{\omega}_{31}.\bar{k}_3 \\ \bar{v}_{A31}.\bar{i}_3 & \bar{v}_{A31}.\bar{j}_3 & 0 \end{vmatrix} \tag{22}$$

Let we substitute radius vector $\bar{r}_{P3}$ from Eg.21 into Eq.1: $\bar{r}_{P1} = \bar{r}_{A31} + \bar{r}_{P3}$ and let us express all vector quantities in the space $\{1\}$

$$\bar{r}_{P1} = \bar{r}_{A31} + \frac{1}{\omega_{31}^{2}} \begin{vmatrix} \bar{i}_1 & \bar{j}_1 & \bar{k}_1 \\ 0 & 0 & \bar{\omega}_{31}.\bar{k}_1 \\ \bar{v}_{A31}.\bar{i}_1 & \bar{v}_{A31}.\bar{j}_1 & 0 \end{vmatrix} \tag{23}$$

then locus of end points of radius vector $\bar{r}_{P1}$ is the fixed centrode k_P (see Fig.3) and Eq.23 is equation of fixed centrode k_P.